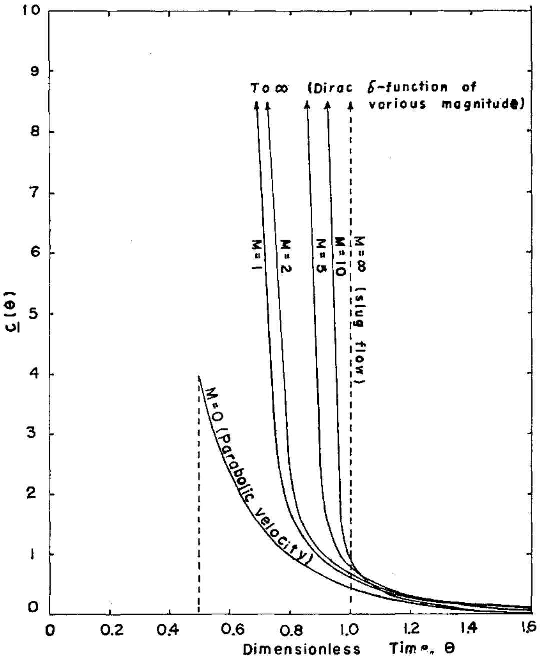

Fig. 1 C(θ) curves for Hartmann flow in a parallel plate channel with M as a parameter.

Since the $\mathbf{C}(\theta)$ function is the derivative of $\mathbf{F}(\theta)$ function with respect to θ, we obtain

$$\left.\begin{aligned} \mathbf{C}(\theta) &= \frac{\bar{u}\sinh M}{(PM/\mu_e{}^2\sigma H_0{}^2)\theta^3} \times \\ &\left\{\frac{1}{\{\cosh M - [\bar{u}\sinh M/(PM/\mu_e{}^2\sigma H_0{}^2)](1/\theta)\}} + \right. \\ &\left.\sum_{j=1}^{\infty}\frac{1\cdot3\cdot5\ldots(2j-1)}{2^j j!\{\cosh M - [\bar{u}\sinh M/(PM/\mu_e{}^2\sigma H_0{}^2)](1/\theta)\}^{2j+1}}\right\} \\ &\qquad \text{for } \theta > \frac{\bar{u}\sinh M}{(PM/\mu_e{}^2\sigma H_0{}^2)(\cosh M - 1)} \\ \mathbf{C}(\theta) &= 0 \text{ for } \theta < \frac{\bar{u}\sinh M}{(PM/\mu_e{}^2\sigma H_0{}^2)(\cosh M - 1)} \end{aligned}\right\} \quad (13)$$

The average velocity of the Hartmann flow, which is an easily measurable quantity, is[5]

$$\bar{u} = (P/\mu_e{}^2\sigma H_0{}^2)(M\cosh M - 1) \quad (14)$$

Substituting this expression into Eq. (13), it is simplified as

$$\left.\begin{aligned} \mathbf{C}(\theta) &= \frac{M\cosh M - \sinh M}{M^2 6^3} \times \\ &\left\{\frac{1}{[\cosh M - (M\cosh M - \sinh M)/M\theta]} + \right. \\ &\left.\sum_{j=1}^{\infty}\frac{1\cdot3\cdot5\ldots(2j-1)}{2^j j![\cosh M - (M\cosh M - \sinh M)/M\theta]^{2j+1}}\right\} \\ &\qquad \text{for } \theta > \frac{M\cosh M - \sinh M}{M} \\ C\theta &= 0 \text{ for } \theta < \frac{M\cosh M - \sinh M}{M} \end{aligned}\right\} \quad (15)$$

The residence time distribution functions for various combinations of the dimensionless Hartmann number M can be calculated from Eq. (15). For illustration, several $\mathbf{C}(\theta)$ curves are plotted on Fig. 1.

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Application of Quasilinearization to Boundary-Layer Equations

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Introduction

The analysis of boundary layers, which exhibit similarity, leads to ordinary differential equations with two-point asymptotic boundary conditions. That is, some boundary conditions are specified at the initial point or wall and others are specified as limits that must be approached at large values of the independent variable corresponding to the edge of the boundary layer.

Because of the inherent nonlinearity of the boundary-layer equations, as well as their considerable complexity for realistic formulations, e.g., compressibility effects, they are usually attacked by numerical integration using digital computers. Since initial conditions are required to start the integration, these must be assumed, based on physical intuition or otherwise. In general, the required boundary conditions at the second point are not satisfied. One method that has been used to find improved boundary conditions is that of making additional solutions with the guessed initial boundary conditions varied in turn followed by linear inverse interpolation (or extrapolation) on the boundary values attained at the second point.[1]

Another method, called successive substitutions or Picard's method, involves solving the differential equation for the highest derivative and substituting the result successively into the right-hand side starting with an approximation that satisfies the boundary conditions.[2] Both of these methods have been used to advantage; however; the first requirers a reasonably good approximation to the boundary conditions and the second to the functions throughout their range. Without sufficiently good starting values, the processes may diverge or converge extremely slowly. In addition, certain other problems arise which can be only named here, such as instability of solutions at large values of the independent variable or large changes of the solution at interior points for small errors in the boundary values.

The purpose of this note is to show by means of a simple example that the quasilinearization method gives promise of producing rapid convergence to solutions of boundary-layer problems from uninspired initial guesses.

Discussion of the Method

Quasilinearization may be viewed as an extension of Newton's method for the solution of algebraic equations to solution of differential equations.[2] Consider the vector equation

$$d\mathbf{X}/dt = \mathbf{g}(\mathbf{X}) \quad (1)$$

where $\mathbf{X}$ is a vector composed of the n-dependant variables in

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the system of equations, t the independant variable, and $\mathbf{g(X)}$ a vector function that gives the derivatives of the dependant variables. The $(k + 1)$ approximation, $\mathbf{X}^{(k+1)}$, to the solution is obtained by expanding the right-hand side about the kth approximation $\mathbf{X}^{(k)}$ and retaining linear terms as follows[3]:

$$\frac{dX_j^{(k+1)}}{dt} = g_j[X^{(k)}] + \sum_{i=1}^{n} \frac{\partial g_j[X^{(k)}]}{\partial X_i} [X_i^{(k+1)} - X_i^{(k)}] \qquad j = 1, 2, \ldots, n \tag{2}$$

where X_i is the ith component of $\mathbf{X}$.

This equation is linear in $\mathbf{X}^{(k+1)}$, and we may use the usual methods for solving linear ordinary differential equations. A particular solution may be obtained by integrating the whole equation starting with the initial conditions

$$X_i(0) = P_i(0) = 0 \qquad i = 1, 2, \ldots, n \tag{3}$$

One homogeneous solution is then made for each of the n boundary conditions B_p with the initial conditions

$$X_i(0) = H_{ij}(0) = \delta_j{}^i = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad i, j = 1, 2, \ldots, n \tag{4}$$

The requirement that the sum of the particular and homogeneous solutions satisfies the boundary conditions determines the combination coefficients C_j:

$$B_p - P_m(t_p) = \sum_{j=1}^{n} C_j H_{mj}(t_p) \qquad p = 1, 2, \ldots, \qquad m = m_p \tag{5}$$

The new approximation $\mathbf{X}^{(k+1)}$ is then generated at each point required by summing the stored values of $\mathbf{P}$ and $\mathbf{H}$,

$$X_i^{(k+1)}(t) = P_i(t) + \sum_{j=1}^{n} C_j H_{ij}(t) \tag{6}$$

or by obtaining the initial conditions in this manner and integrating forward.

If the particular solution $\mathbf{P}$ is required to satisfy the N known initial conditions, then there are only $n - N = n_2$ remaining boundary conditions to satisfy, and, in Eq. (5), only n_2 homogeneous solutions are required.[4] All of the foregoing equations remain unchanged except that the ranges of subscripts j and p are reduced to 1 to n_2. The number of equations to be integrated is reduced from $n + 1$ to $(n/2) + 1$ or less, since at least half the boundary conditions must be known at one of the ends of a two point boundary-value problem. This is an important saving in computation.

Sample Boundary-Layer Solution

The Falkner and Skan[5] equations describing the flow of an incompressible fluid boundary layer over two-dimensional wedges was chosen as a sample problem for the quasilinearization method because it exhibits many of the features such as nonlinearity and instability, characteristic of many of the more complex systems of boundary-layer equations.

The equation is

$$f''' + ff'' + \beta(1 - f'^2) = 0 \tag{7}$$

with the boundary conditions $\eta = 0{:}f = 0, f' = 0$; $\eta = \eta_{\text{edge}}{:}f' = 1$. Primes denote differentiation with respect to a wall distance parameter η defined by

$$\eta = y\left[\frac{(m+1)U}{2}\right]^{1/2} x^{(m-1)/2} \tag{8}$$

The dependant variable f is related to the usual stream function ψ by

$$\psi(x, \eta) = \left(\frac{2\nu u_1}{m + 1}\right)^{1/2} x^{(m+1)/2} f(\eta) \tag{9}$$

The included angle of the wedge β is related to the exponent m of the potential velocity U by

$$\beta = 2m/(m + 1) \tag{10}$$

where

$$U(x) = u_1 x^m \tag{11}$$

In the numerical experiment below, β was taken equal to unity corresponding to a two-dimensional stagnation point.

Let us now establish the correspondence between the general method and our sample equation. The independent variable corresponds to t, the equation

$$\begin{bmatrix} (f'')' \\ (f')' \\ (f)' \end{bmatrix} = \begin{bmatrix} -ff'' + \beta(1 - f'^2) \\ f'' \\ f' \end{bmatrix} \tag{12}$$

corresponds to Eq. (1), and the Jacobian matrix $[\partial g_i/\partial x_i]$ is given by

$$\left[\frac{\partial g_j}{\partial X_i}\right] = \begin{bmatrix} -f & 2\beta f' & -f'' \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{13}$$

The boundary conditions are specified by $[B_p]$, $[t_p]$, $[m_p]$:

$$[B_p] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad [t_p] = \begin{bmatrix} 0 \\ \eta \text{ edge} \\ \eta \text{ edge} \end{bmatrix} \qquad [m_p] = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \tag{14}$$

Comments on the Computer Solution

The equations in the last section were programed in FORTRAN II, and the solution was carried out on an IBM 7094 computer. An Adams-Moulton integration subroutine NTGRAT[6] was used which had a variable integration step size and a larger fixed printing interval. The current approximation $\mathbf{X}^{(k)}$, as well as $\mathbf{P}$ and the $\mathbf{H}_j$, generated by integration, were stored at the fixed print interval. Intermediate points were obtained by Sterling interpolation[7] truncated with second differences.

The n $(n_2 + 1)$ equations for $\mathbf{P}$ and the $\mathbf{H}_j$ were integrated simultaneously, requiring only one evaluation of g_i and $\partial g_i/\partial \mathbf{X}_i$ at each value of η. After the integration was terminated at the outer edge of the boundary layer, the combination coefficients C_j were evaluated from Eq. (5), and the stored values substituted into Eq. (6) to obtain the new approximation $\mathbf{X}^{(k+1)}$.

Computation was saved by relaxing truncation error requirements where possible. At each stage, the best available initial conditions were used for $\mathbf{P}$, and the contributions of the $\mathbf{H}_j$ to $\mathbf{X}^{(k+1)}$ diminished as the process converged. Thus, the relative accuracy required of the $\mathbf{H}_j$ was not as large as that of $\mathbf{P}$. Likewise, the integrations in the early iterations of the process were not carried out as accurately as those in the later iterations.

Convergence of the quasilinearization process was tested by comparing the values of $X_i^{(k)}$ and $X_i^{(k+1)}$ at the print interval.

Results

The first (initial), second, third, and fifth (converged) profiles of f'', f', and f are shown in Fig. 1. The initial approximation is certainly uninspired. The second approximation overshoots the final curve, and succeeding approximations converge rapidly from that side. Note that the largest proportional deviations in the second approximation are found in the center of the f' (velocity) profile, since both ends are forced to satisfy the boundary conditions by the process of adding solutions. The large curvatures in $f'^{(2)}$ and $f''^{(2)}$ produce strange bumps in $f^{(3)}$ which disappear in later ap-

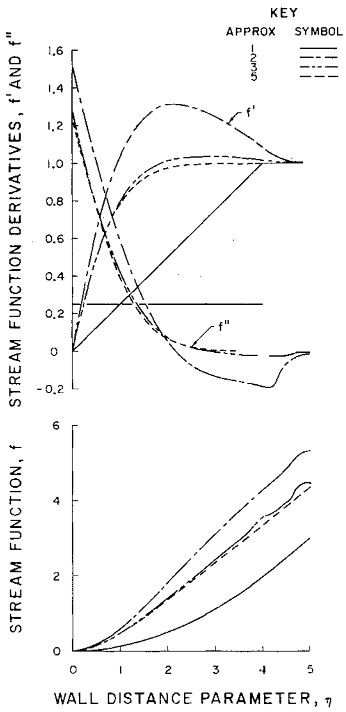

Fig. 1 Quasilinearization solution for the boundary layer on a two-dimensional stagnation point, convergence of approximations for stream function f and its derivatives.

proximations. These types of behavior are probably typical of the convergence process.

No instability is shown by the final profiles, although the velocity comes into its asymptotic value of unity at about 2.5, and the solution was carried out to an η of 5. The "shoot and correct" method of successive approximations mentioned in the Introduction, however, appears to be very sensitive to instability.

The values of all functions in the fifth approximation differed from the corresponding functions in the fourth approximation at each of the 26 tabulated points by less than 5 in the third decimal place. The value of f'' (shear stress) in the fifth approximation was 1.2397 compared with the published value[8] of 1.2326. This accuracy is consistent with the truncation error of 0.001 specified to the integration subroutine. The use of a tighter truncation error bound or convergence criterion (more cycles) and the concomitant additional computer time did not seem justified for the purpose of this test case.

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Orbit Determination by Angular Measurements

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A NEW method is here introduced for the preliminary determination of satellite orbits about the earth. Let $\mathbf{s}$, $\rho\mathbf{p}$ denote the position vectors of the geocenter and an orbiting satellite relative to a tracking station at time t, in which ρ, $\mathbf{p}$ denote slant range and unit vector, respectively. The position of the satellite relative to a geocentered inertial frame is therefore defined by the vector equation $\mathbf{r} = \rho\mathbf{p} - \mathbf{s}$.

Let vector and scalar functions of t_i, $i = 1, 2, 3$, be denoted simply by use of the subscript i, so that $\mathbf{s}(t_i) \triangleq \mathbf{s}_i$, $\rho(t_i) \triangleq \rho_i$. The time intervals will be denoted by Δ_1, Δ_2, Δ_3, so that $\mathbf{r}_2 = \mathbf{r}(t_1 + \Delta_1)$, $\mathbf{r}_3 = \mathbf{r}(t_2 + \Delta_2)$. Let B_{ij} denote the area of the triangle bounded by the vectors $\mathbf{r}_i$, $\mathbf{r}_j$, $\mathbf{r}_j - \mathbf{r}_i$. It follows that $|\mathbf{r}_i \times \mathbf{r}_j| = 2B_{ij}$. Consequently,

$$\frac{\mathbf{r}_1 \times \mathbf{r}_2}{B_{12}} = \frac{\mathbf{r}_2 \times \mathbf{r}_3}{B_{23}} = \frac{\mathbf{r}_1 \times \mathbf{r}_3}{B_{13}} \tag{1}$$

The ratios of the triangular areas are expressible by the formulas of Gibbs (see Ref. 1) which, in the present terminology, are given by

$$\frac{B_{23}}{B_{13}} = \frac{\Delta_2}{\Delta_3}\left[1 + \frac{\mu\Delta_1}{6r_2^3}(\Delta_2 + \Delta_3) + \frac{\mu}{12} \times \left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right)(\Delta_1\Delta_3 - \Delta_2^2) + \ldots\right]$$

$$\frac{B_{12}}{B_{13}} = \frac{\Delta_1}{\Delta_3}\left[1 + \frac{\mu\Delta_2}{6r_2^3}(\Delta_1 + \Delta_3) - \frac{\mu}{12} \times \left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right)\left(\frac{\Delta_2}{\Delta_1}\right)(\Delta_2\Delta_3 - \Delta_1^2) + \ldots\right] \tag{2}$$

The measurements of the azimuth and elevation of a satellite at three instants of time t_1, t_2, t_3 provide sufficient tracking data for a preliminary determination of the orbit of the satellite about the earth. For such a determination, it is convenient to express inertial components of the tracking station position vectors $-\mathbf{s}_i$ $(i = 1, 2, 3)$ in terms of the radius of the earth, the station's colatitude and longitude, and the angles of the earth's rotation at times t_i, $(i = 1, 2, 3)$. The line-of-sight vectors $\mathbf{p}_i$ $(i = 1, 2, 3)$ are computed from the geometry relating two local station-centered reference frames, namely, that in which azimuth and elevation are direction coordinates and that whose axes are parallel to those of the geocentered inertial Cartesian frame. The transformation equations will be omitted here; the essential point of the present discussion is that the components of the vectors $\mathbf{p}_i$, $\mathbf{s}_i$ $(i = 1, 2, 3)$ will be assumed known. Since the positions of the satellite are defined by $\mathbf{r}_i = \rho_i\mathbf{p}_i - \mathbf{s}_i$, $i = 1, 2, 3$, only the slant ranges ρ_i have to be calculated to complete the determination of $\mathbf{r}_i$. An iterative process will be described which has been devised for this purpose. For other iterative schemes of the general nature used in this note, see Herget.[2]

The vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ are linearly related by the equation

$$B_{23}\mathbf{r}_1 - B_{13}\mathbf{r}_2 + B_{12}\mathbf{r}_3 = 0 \tag{3}$$

In fact, vector multiplications of Eq. (3), first by $\mathbf{r}_2$ and then by $\mathbf{r}_3$, yield Eqs. (1). Conversely, Eqs. (1) imply that the vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ are coplanar and determine the equation of the

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